Chapter 3 • Introduction to Calculus

Review of Prerequisite Skills

2. **e.** The slope of line is

$$m = \frac{12-6}{4-(-1)}$$
$$= \frac{6}{5}$$

The equation of the line is in the form $y - y_1 = m(x - x_1)$. The point is $(-1, 6)$ and $m = \frac{6}{5}$.

The equation of the line is $y - 6 = \frac{6}{5}(x+1)$ or $6x - 5y + 36 = 0$.

4. $f(x) = \begin{cases} \sqrt{3-x} & \text{if } x < 0 \\ \sqrt{3+x} & \text{if } x \geq 0 \end{cases}$

a. $f(-33) = 6$

b. $f(0) = \sqrt{3}$

c. $f(78) = 9$

6. **b.** $\frac{6+\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}+\sqrt{6}}{3}$

d. $\frac{1}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}}$

$$= \frac{3-\sqrt{3}}{9-3}$$
$$= \frac{3-\sqrt{3}}{6}$$

g. $\frac{5\sqrt{3}}{2\sqrt{3}+4} \times \frac{2\sqrt{3}-4}{2\sqrt{3}-4}$

$$= \frac{30-20\sqrt{3}}{12-16}$$
$$= \frac{30-20\sqrt{3}}{-4}$$
$$= \frac{10\sqrt{3}-15}{2}$$

7. **b.** $\frac{\sqrt{3}}{6+\sqrt{2}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3}{6\sqrt{3}+\sqrt{6}}$

c. $\frac{\sqrt{7}-4}{5} \times \frac{\sqrt{7}+4}{\sqrt{7}+4}$

$$= \frac{7-16}{5\sqrt{7}+20} = \frac{-9}{5\sqrt{7}+20}$$

d. $\frac{2\sqrt{3}-5}{3\sqrt{2}} \times \frac{2\sqrt{3}+5}{2\sqrt{3}+5}$

$$= \frac{12-25}{6\sqrt{6}+15\sqrt{2}} = \frac{-13}{6\sqrt{6}+15\sqrt{2}}$$

f. $\frac{2\sqrt{3}+\sqrt{7}}{5} \times \frac{2\sqrt{3}-\sqrt{7}}{2\sqrt{3}-\sqrt{7}}$

$$= \frac{12-7}{10\sqrt{3}-5\sqrt{7}} = \frac{5}{10\sqrt{3}-5\sqrt{7}}$$

8. **h.** $x^3 - 2x^2 + 3x - 6 = x^2(x-2) + 3(x-2)$

$$= (x-2)(x^2+3)$$

i. $2x^3 - x^2 - 7x + 6$

$$f(1) = 2 - 1 - 7 + 6$$
$$= 0$$

Therefore, $x - 1$ is a factor.
By division, the other factor is $2x^2 + x - 6$.
Therefore, $2x^3 - x^2 - 7x + 6 = (x-1)(2x^2+x-6)$

$$= (x-1)(2x-3)(x+2).$$

9. **j.** $y = \frac{7}{x^2 - 3x - 4}$

$$= \frac{7}{(x-4)(x+1)}$$

The domain is $x \in R, x \neq 4$, or -1.

k. $y = \frac{6x}{2x^2 - 5x - 3}$

$$= \frac{6x}{(2x+1)(x-3)}$$

The domain is $x \in R, x = -\frac{1}{2}$, and 3.

Section 3.1

Investigation 1

1. $y = x^2$

a. $Q_1(3.5, 12.25)$ **b.** $Q_2(3.1, 9.61)$
c. $Q_3(3.01, 9.0601)$ **d.** $Q_4(3.001, 9.00600)$

2. Slope of secant $P(3, 9)$, Q_i:

$PQ_1 \to 6.5$
$PQ_2 \to 6.1$
$PQ_3 \to 6.01$
$PQ_4 \to 6.001$

3. **a.** $Q_5(2.5, 6.25)$ **b.** $Q_6(2.9, 8.41)$
c. $Q_7(2.99, 8.9401)$ **d.** $Q_8(2.999, 8.994001)$

$PQ_5 \to 5.5$
$PQ_6 \to 5.9$
$PQ_7 \to 5.99$
$PQ_8 \to 5.999$

4. Slope of the tangent at $P(3, 9)$ seems to be 6.

Investigation 2

3. $P(1,\ 1)$, $Q_1(1.5, f(1.5))$

Slope $PQ_1 \to 2.5$
Slope $PQ_2(1.1, f(1.1)) = 2.1$
Slope $PQ_3(1.01, f(1.01)) = 2.01$
Slope $PQ_4(1.001, f(1.001)) = 2.001$
Slope $PQ_5(1.0001, f(1.0001)) = 2.0001$

5. Slope of the tangent at $P(1, 1)$ is 2.

Investigation 3

1. $P(3, 9)\ \ Q\left(3+h,\ (3+h)^2\right)$

$$\text{Slope of PQ } = \frac{(3+h)^2 - 9}{3+h-3} = \frac{9+6h+h^2-9}{h} = 6+h, \quad h \neq 0$$

2. Substitute $h = 0$ in the above slope.

Exercise 3.1

4. **b.**
$$\frac{(5+h)^3 - 125}{h} = \frac{(5+h-5)\left((5+h)^2 + 5(5+h) + 25\right)}{h} = \frac{h(75+15h+h^2)}{h} = 75+15h+h^2$$

c.
$$\frac{(3+h)^4 - 81}{h} = \frac{\left((3+h)^2 - 9\right)\left((3+h)^2 + 9\right)}{h} = \frac{(9+6h+h^2-9)(9+6h+h^2+9)}{h} = (6+h)(18+6h+h^2) = 108+54h+12h^2+h^3$$

d.
$$\frac{\frac{1}{1+h} - 1}{h} = \frac{1-1-h}{h(1+h)} = -\frac{1}{1+h}$$

e.
$$\frac{3(1+h)^2 - 3}{h} = \frac{3\left((1+h)^2 - 1\right)}{h} = \frac{3(1+2h+h^2-1)}{h} = \frac{3(2h+h^2)}{h} = 6+3h$$

f.
$$\frac{(2+h)^3 - 8}{h} = \frac{(2+h-2)\left((2+h)^2 + 2(2+h) + 4\right)}{h} = 12+6h+h^2$$

g.
$$\frac{\frac{3}{4+h} - \frac{3}{4}}{h} = \frac{\frac{12-12-3h}{4(4h)}}{h} = \frac{-3}{4(4+h)}$$

5. **a.**
$$\frac{\sqrt{16+h} - 4}{h} = \frac{16+h-16}{h\left(\sqrt{16+h} + 4\right)} = \frac{1}{\sqrt{16+h} + 4}$$

b.
$$\frac{\sqrt{h^2+5h+4} - 2}{h} = \frac{h^2+5h+4-4}{h\left(\sqrt{h^2+5h+4} + 2\right)} = \frac{h+5}{\sqrt{h^2+5h+4} + 2}$$

c. $\dfrac{\sqrt{5+h}-\sqrt{5}}{h}=\dfrac{5+h-5}{h\left(\sqrt{5+h}+\sqrt{5}\right)}$

$=\dfrac{1}{\sqrt{5+h}+\sqrt{5}}$

6. **a.** $P(1,3),\ Q(1+h, f(1+h)),\ f(x)=3x^2$

$$m=\frac{3(1+h)^2-3}{h}$$
$$=6+3h$$

b. $R(1,3),\ S\left(1+h,\ (1+h)^3+2\right)$

$$m=\frac{(1+h)^3+2-3}{h}$$
$$=\frac{1+3h+3h^2+h^3-1}{h}$$
$$=3+3h+h^2$$

c. $T(9,\ 3),\ U\left(9+h,\ \sqrt{9+h}\right)$

$$m=\frac{\sqrt{9+h}-3}{h}\bullet\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$$
$$=\frac{1}{\sqrt{9+h}+3}$$

7. **a.**

P	Q	Slope
$(2, 8)$	$(3, 27)$	19
↓	$(2.5, 15.625)$	15.25
	$(2.1, 9.261)$	12.61
	$(2.01, 8.120601)$	12.0601
	$(1, 1)$	7
	$(1.5, 3.375)$	9.25
	$(1.9, 6.859)$	11.41
	$(1.99, 7.880599)$	11.9401

8. **a.** $y=3x^2,\ (-2, 12)$

$$m=\lim_{h\to 0}\frac{3(-2+h)^2-12}{h}$$
$$=\lim_{h\to 0}\frac{12-12h+3h^2-12}{h}$$
$$=\lim_{h\to 0}(-12+3h)$$
$$=-12$$

b. $y=x^2-x$ at $x=3,\ y=6$

$$m=\lim_{h\to 0}\frac{(3+h)^2-(3+h)-6}{h}$$
$$=\lim_{h\to 0}\frac{9+6h+h^2-3-h-6}{h}$$
$$=\lim_{h\to 0}(5+h)$$
$$=5$$

c. $y=x^3$ at $x=-2,\ y=-8$

$$m=\lim_{h\to 0}\frac{(-2+h)^3+8}{h}$$
$$=\lim_{h\to 0}\frac{-8+12h-6h^2+h^3+8}{h}$$
$$=\lim_{h\to 0}(12-6h+h^2)$$
$$=12$$

9. **a.** $y=\sqrt{x-2}$; (3, 1)

$$m=\lim_{h\to 0}\frac{\sqrt{3+h-2}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\times\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}$$
$$=\frac{1}{2}$$

b. $y=\sqrt{x-5}$ at $x=9,\ y=2$

$$m=\lim_{h\to 0}\frac{\sqrt{9+h-5}-2}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\times\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}$$
$$=\frac{1}{4}$$

c. $y=\sqrt{5x-1}$ at $x=2,\ y=3$

$$m=\lim_{h\to0}\frac{\sqrt{10+5h-1}-3}{h}$$
$$=\lim_{h\to0}\frac{\sqrt{9+5h}-3}{h}\times\frac{\sqrt{9+5h}+3}{\sqrt{9+5h}+3}$$
$$=\lim_{h\to0}\frac{5}{\sqrt{9+5h}+3}$$
$$=\frac{5}{6}$$

10. a. $y=\dfrac{8}{x}$ at (2, 4)

$$\lim_{x\to5}\frac{|x-5|}{x-5}\ m=\lim_{h\to0}\frac{\frac{8}{2+h}-4}{h}$$
$$=\lim_{h\to0}\frac{-4}{2+h}$$
$$=-2$$

b. $y=\dfrac{8}{3+x}$ at $x=1;\ y=2$

$$m=\lim_{h\to0}\frac{\frac{8}{4+h}-2}{h}$$
$$=\lim_{h\to0}\frac{-2}{4+h}$$
$$=-\frac{1}{2}$$

c. $y=\dfrac{1}{x+2}$ at $x=3;\ y=\dfrac{1}{5}$

$$m=\lim_{h\to0}\frac{\frac{1}{5+h}-\frac{1}{5}}{h}$$
$$=\lim_{h\to0}\frac{-1}{5(5+h)}$$
$$=-\frac{1}{10}$$

11. a. $y=x^2-3x,\ (2,-2);\ y'=2x-3,\ m=1$

b. $f(x)(-2),-2;=\dfrac{4}{x},\quad y'=-\dfrac{4}{x^2},\ m=-1$

c. $y=3x^3$ at $x=1;\ y'=9x^2,\ m=9$

d. $y=\sqrt{x-7}$ at $x=16;\ y'=\dfrac{1}{2}(x-7)^{-\frac{1}{2}},\ m=\dfrac{1}{6}$

e. $f(x)=\sqrt{16-x},\ y=5;\ x=-9,\ y'=1-\dfrac{1}{2}(16-x)^{-\frac{1}{2}},$
$m=-\dfrac{1}{10}$

f. $y=\sqrt{25-x^2},\ (3,4);\ y'=\dfrac{1-2x}{2\sqrt{25-x^2}},\ m=-\dfrac{3}{4}$

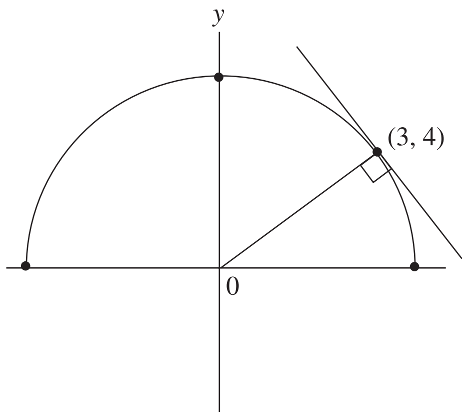
g. $y=\dfrac{4+x}{x-2}$ at $x=8;\ y=2$

$$m=\lim_{h\to0}\frac{\frac{12+h}{6+h}-2}{h}$$
$$=\lim_{h\to0}\frac{-h}{h(6+h)}$$
$$=-\frac{1}{6}$$

h. $y=\dfrac{8}{\sqrt{x+11}}$ at $x=5;\ y=2$

$$m=\lim_{h\to0}\frac{\frac{8}{\sqrt{16+h}}-2}{h}$$
$$=\lim_{h\to0}\frac{8-2\sqrt{16+h}}{h\sqrt{16+h}}\times\frac{4+\sqrt{16+h}}{4+\sqrt{16+h}}$$
$$=2\lim_{h\to0}\frac{16-16-h}{h\sqrt{16+h}\left(4+\sqrt{16+h}\right)}$$
$$=2\bullet\frac{-1}{4(8)}$$
$$=-\frac{1}{16}$$

12.



$y = \sqrt{25 - x^2} \rightarrow$ Semi-circle centre (0, 0) rad 5, $y \geq 0$

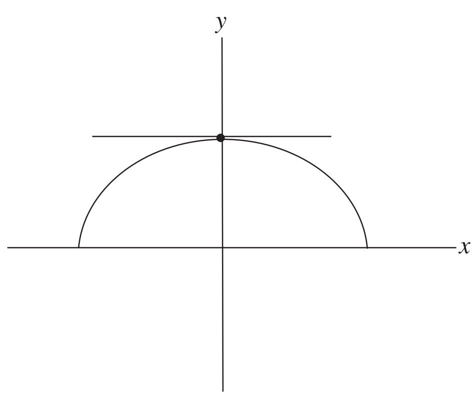
OA is a radius.

The slope of OA is $\frac{4}{3}$.

The slope of the tangent is $-\frac{3}{4}$.

13. Take values of x close to the point, then determine $\frac{\Delta y}{\Delta x}$.

14.



Since the tangent is horizontal, the slope is 0.

16. $D(p) = \frac{20}{\sqrt{p-1}}$, $p > 1$ at (5, 10)

$$m = \lim_{h \to 0} \frac{\frac{20}{\sqrt{4+h}} - 10}{h}$$

$$= 10 \lim_{h \to 0} \frac{2 - \sqrt{4+h}}{h\sqrt{4+h}} \times \frac{2 + \sqrt{4+h}}{2 + \sqrt{4+h}}$$

$$= 10 \lim_{h \to 0} \frac{4 - 4 - h}{h\sqrt{4+h}\left(2 + \sqrt{4+h}\right)}$$

$$= -\frac{10}{8}$$

$$= -\frac{5}{4}$$

17. $C(t) = 100t^2 + 400t + 5000$

Slope at $t = 6$

$C'(t) = 200t + 400$

$C'(6) = 1200 + 400 = 1600$

Increasing at a rate of 1600 papers per month.

18. Point on $f(x) = 3x^2 - 4x$ tangent parallel to $y = 8x$.

Therefore, tangent line has slope 8.

$$\therefore m = \lim_{h \to 0} \frac{3(h+a)^2 - 4(h+a) - 3(a^2 + 4a)}{h} = 8$$

$$\lim_{h \to 0} \frac{3h^2 + 6ah - 4h}{h} = 8$$

$$\therefore 6a - 4 = 8$$

$$a = 2$$

The point has coordinates (2, 4).

19. $y = \frac{1}{3}x^3 - 5x - \frac{4}{x}$

$$\frac{1}{3}(a+h)^3 - \frac{1}{3}a^3$$

$$= a^2h + ah^2 + \frac{1}{3}h^3$$

$$\lim_{h \to 0}\left(a^2 + ah + \frac{1}{3}h^3\right) = a^2$$

$$5\lim_{h \to 0} - \frac{(a+h) - (-a)}{h} = -5$$

$$-\frac{4}{a+h} + \frac{4}{a} = -\frac{4a + 4a + 4h}{a(a+h)}$$

$$\lim_{h \to 0} \frac{4}{a(a+h)} = \frac{4}{a^2}$$

$$m = a^2 - 5 + \frac{4}{a^2} = 0$$

$$a^4 - 5a^2 + 4 = 0$$

$$(a^2 - 4)(a^2 - 1) = 0$$

$$a = \pm 2, \ a = \pm 1$$

Points on the graph for horizontal tangents are:

$$\left(-2, \frac{28}{3}\right), \left(-1, \frac{26}{3}\right), \left(1, -\frac{26}{3}\right), \left(2, -\frac{28}{3}\right).$$

20. $y = x^2$ and $y = \frac{1}{2} - x^2$

$$x^2 = \frac{1}{2} - x^2$$

$$x^2 = \frac{1}{4}$$

$$x = \frac{1}{2} \text{ or } x = -\frac{1}{2}$$

The points of intersection are

$P\left(\frac{1}{2}, \frac{1}{4}\right)$, $Q\left(-\frac{1}{2}, \frac{1}{4}\right)$.

Tangent to $y = x$:

$$\begin{aligned} m &= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h\to 0} \frac{2ah + h^2}{h} \\ &= 2a \end{aligned}$$

The slope of the tangent at $a = \frac{1}{2}$ is $1 = m_p$,

at $a = -\frac{1}{2}$ is $n - 1 = m_q$.

Tangents to $y = \frac{1}{2} - x^2$:

$$\begin{aligned} m &= \lim_{h\to 0} \frac{\left[\frac{1}{2} - (a+h)^2\right] - \left[\frac{1}{2} - a^2\right]}{h} \\ &= \lim_{h\to 0} \frac{-2ah - h^2}{h} \\ &= -2a. \end{aligned}$$

The slope of the tangents at $a = \frac{1}{2}$ is $-1 = M_p$;

at $a = -\frac{1}{2}$ is $1 = M_q$

$m_p M_p = -1$ and $m_q M_q = -1$

Therefore, the tangents are perpendicular at the points of intersection.

Exercise 3.2

2. **a.** $\frac{s(9) - s(2)}{7}$. Slope of the secant between the points $(2, s(2))$ and $(9, s(9))$.

b. $\lim_{h\to 0} \frac{s(6+h) - s(6)}{h}$. Slope of the tangent at the point $(6, s(6))$.

3. $\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$. Slope of the tangent to the function with equation $y = \sqrt{x}$ at the point (4, 2).

7. $s(t) = 5t^2, 0 \le t \le 8$

a. Average velocity during the first second:

$$\frac{s(1) - s(0)}{1} = 5 \text{ m/s};$$

third second:

$$\frac{s(3) - s(2)}{1} = \frac{45 - 20}{1} = 25 \text{ m/s};$$

eighth second:

$$\frac{s(8) - s(7)}{1} = \frac{320 - 245}{1} = 75 \text{ m/s}.$$

b. Average velocity $3 \le t \le 8$

$$\begin{aligned} \frac{s(8) - s(3)}{8 - 3} &= \frac{320 - 45}{5} \\ &= \frac{275}{5} \\ &= 55 \text{ m/s} \end{aligned}$$

c. $s(t) = 320 - 5t^2$

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{-5(2+h)^2 + 5(2)^2}{h} \\ &= 5\lim_{h\to 0} \frac{-4h + h^2}{h} \\ &= -20 \end{aligned}$$

Velocity at $t = 2$ is 20 m/s downward.

8. $s(t) = 8t(t+2), 0 \le t \le 5$

$s(t) = \lim_1 \; t - \text{hours}$

a. i) from $t = 3$ to $t = 4$

Average velocity $\dfrac{s(4) - s(3)}{1}$

$= 32(6) - 24(5)$

$= 24(8 - 5)$

$= 72 \text{ km/h}$

ii) from to $t = 3.1$

$\dfrac{s(3.1) - s(3)}{0.1}$

$= \dfrac{126.48 - 120}{0.1}$

$= 64.8 \text{ km/h}$

iii) $3 \le t \le 3.01$

$\dfrac{s(3.01) - s(3)}{0.01}$

$= 64.08 \text{ km/h}$

b. Instantaneous velocity is approximately 64 km/h.

c. At $t = 3$.

$s(t) = 8t^2 + 16t$

$v(t) = 16t + 16$

$v(3) = 48 + 16$

$= 64 \text{ km/h}$

9. a. $N(t) = 20t - t^2$

$\dfrac{N(3) - N(2)}{1}$

$= \dfrac{51 - 36}{1}$

$= 15$

15 terms are learned between $t = 2$ and $t = 3$.

b. $N'(t) = 20 - 2t$

$N'(2) = 20 - 4 = 16$

At $t = 2$, the student is learning at a rate of 16 terms/hour.

10. a. M in mg in 1 mL of blood t hours after the injection.

$M(t) = -\dfrac{1}{3}t^2 + t, \; 0 \le t \le 3$

$M(t) = -\dfrac{2}{3}t + 1$

$M(2) = -\dfrac{4}{3} + 1 = -\dfrac{1}{3}$

Rate of change is $-\dfrac{1}{3}$ mg/h.

b. Amount of medicine in 1 mL of blood is being dissipated throughout the system.

11. $t = \sqrt{\dfrac{s}{5}}$

$t' = \dfrac{1}{2}\left(\dfrac{s}{5}\right)^{-\frac{1}{2}} \bullet \dfrac{1}{5}$

$= \dfrac{1}{10} \bullet \left(\sqrt{\dfrac{s}{5}}\right)^{-1}$

$s = 125, \; t' = \dfrac{1}{10} \bullet \dfrac{1}{5} = \dfrac{1}{50}$

At $s = 125$, rate of change of time with respect to height is $\dfrac{1}{50}$ s/m.

12. $T(h) = \dfrac{60}{h+2}$

$T'(h) = -(60)(h+2)^{-2}$

$= -\dfrac{60}{(h+2)^2}$

$T'(3) = -\dfrac{60}{25}$

$= -\dfrac{12}{5}$

Temperature is decreasing at $\dfrac{12}{5}^{\circ}$ C/km.

13. $h = 25t^2 - 100t + 100$

$h(t)' = 50t - 100$

When $h = 0$, $25t^2 - 100t + 100 = 0$

$t^2 - 4t + 4 = 0$

$(t-2)^2 = 0$

$t = 2.$

$h'(2) = 0$

It hit the ground in 2 s at a speed of 0 m/s.

14. Sale of x balls per week:

$P(x) = 160x - x^2$ dollars.

a. $P(40) = 160(40) - (40)^2$

$= 4800$

Profit on the sale of 40 balls is \$4800.

b. $P'(x) = 160 - 2x$

$P'(40) = 160 - 80$

$= 80$

Rate of change of profit is \$80 per ball.

c. $160 - 2x > 0$

$-2x > 160$

$x < 80$

Rate of change of profit is positive when the sales level is less than 80.

15. a. $f(x) = -x^2 + 2x + 3;\ (-2, -5)$

$$\lim_{x\to-2}\frac{f(x)-f(-2)}{x+2}$$

$$= \lim_{x\to-2}\frac{-x^2+2x+3+5}{x+2}$$

$$= \lim_{x\to-2}\frac{-(x^2-2x-8)}{x+2}$$

$$= -\lim_{x\to-2}\frac{(x-4)(x+2)}{x+2}$$

$$= -\lim_{x\to-2}(x-4)$$

$$= 6$$

b. $f(x) = \frac{x}{x-1},\ x = 2$

$$\lim_{x\to2}\frac{\frac{x}{x-1}-2}{x-2}$$

$$= \lim_{x\to2}\frac{x-2x+2}{(x-1)(x-2)}$$

$$= \lim_{x\to2}\frac{-(x-2)}{(x-1)(x-2)}$$

$$= -1$$

c. $f(x) = \sqrt{x+1},\ x = 24$

$$\lim_{x\to24}\frac{f(x)-f(24)}{x-24}$$

$$= \lim_{x\to24}\frac{\sqrt{x+1}-5}{x-24}\bullet\frac{\sqrt{x+1}+5}{\sqrt{x+1}+5}$$

$$= \lim_{x\to24}\frac{x-24}{(x-24)\left(\sqrt{x+1}+5\right)}$$

$$= \frac{1}{10}$$

17. $C(x) = F + V(x)$

$C(x+h) = F + V(x+h)$

Rate of change of cost is

$$\lim_{x\to R}\frac{C(x+h)-C(x)}{h}$$

$$= \lim_{x\to h}\frac{V(x+h)-V(x)}{h},$$

which is independent of F – (fixed costs).

18. $P(r) = \pi r^2$

Rate of change of area is

$$\lim_{h\to0}\frac{A(r+h)-A(r)}{h}$$

$$= \lim_{h\to0}\frac{\pi(r+h)^2-\pi r^2}{h}$$

$$= \pi\lim_{h\to0}\frac{(r+h-r)(r+h+r)}{h}$$

$$= 2\pi r$$

$r = 100$ m

Rate is 200π m^2/m.

19. Cube of dimension x by x by x has volume $V = x^3$. Surface area is $6x^2$.

$$V'(x) = 3x^2 = \frac{1}{2} \text{ surface area.}$$

Exercise 3.3

13. $f(x) = mx + b$

$$\lim_{x\to 1} f(x) = -2 \quad \therefore m + b = -2$$

$$\lim_{x\to -1} f(x) = 4 \quad \therefore -m + b = 4$$

$$2b = 2$$

$$b = 1, \quad m = -3$$

14. $f(x) = ax^2 + bx + c, \ a \neq 0$

$$f(0) = 0 \quad \therefore c = 0$$

$$\lim_{x\to 1} f(x) = 5 \quad \therefore a + b = 5$$

$$\lim_{x\to -2} f(x) = 8 \quad \therefore 4a - 2b = 8$$

$$6a = 18$$

$$a = 3, \ b = 2$$

Therefore, the values are $a = 3, \ b = 2,$ and $c = 0$.

Exercise 3.4

7. a. $$\lim_{x\to 2} \frac{4 - x^2}{2 - x} = \lim_{x\to 2} \frac{(2-x)(2+x)}{(2-x)}$$

$$= \lim_{x\to 2}(2 + x)$$

$$= 4$$

b. $$\lim_{x\to -2} \frac{4 - x^2}{2 + x} = \lim_{x\to -2} \frac{(2-x)(2+x)}{(2+x)}$$

$$= 4$$

c. $$\lim_{x\to 0} \frac{7x - x^2}{x} = \lim_{x\to 0} \frac{x(7-x)}{x} = 7$$

d. $$\lim_{x\to -1} \frac{2x^2 + 5x + 3}{x + 1} = \lim_{x\to -1} \frac{(x+1)(2x+3)}{x+1} = 5$$

e. $$\lim_{x\to -\frac{4}{3}} \frac{3x^2 + x - 4}{3x + 4} = \lim_{x\to -\frac{4}{3}} \frac{(3x+4)(x-1)}{3x+4} = -\frac{7}{3}$$

f. $$\lim_{x\to 3} \frac{x^3 - 27}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x^2 + 3x + 9)}{x - 3}$$

$$= 9 + 9 + 9 = 27$$

g. $$x^3 + 2x^2 - 4x - 8 = x^2(x + 2) - 4(x + 2)$$

$$= (x - 2)(x + 2)(x + 2)$$

$$\lim_{x\to -2} \frac{x^3 + 2x^2 - 4x - 8}{x + 2} = \lim_{x\to -2} \frac{(x-2)(x+2)(x+2)}{(x+2)}$$

$$= 0$$

h. $$2x^3 - 5x^2 + 3x - 2 = (x - 2)(2x^2 - x + 1)$$

$$\lim_{x\to 2} \frac{2x^3 - 5x^2 + 3x - 2}{2(x - 2)} = \frac{7}{2}$$

i. $$\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x} \times \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} = \lim_{x\to 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{2}$$

j. $$\lim_{x\to 0} \frac{2 - \sqrt{4+x}}{x} \times \frac{2 + \sqrt{4+x}}{2 + \sqrt{4+x}}$$

$$= \lim_{x\to 0} \frac{-1}{2 + \sqrt{4+x}} = -\frac{1}{4}$$

k. $$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x\to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)}$$

$$= \frac{1}{4}$$

l. $$\lim_{x\to 0} \frac{\sqrt{7-x} - \sqrt{7+x}}{x} \times \frac{\sqrt{7-x} + \sqrt{7+x}}{\sqrt{7-x} + \sqrt{7+x}}$$

$$= \lim_{x\to 0} \frac{7 - x - 7 - x}{x(\sqrt{7-x} + \sqrt{7+x})}$$

$$= -\frac{1}{\sqrt{7}}$$

m. $\lim_{x\to 1}\frac{\sqrt{5-x}-\sqrt{3+x}}{x-1}\times\frac{\sqrt{5-x}+\sqrt{3+x}}{\sqrt{5-x}+\sqrt{3+x}}$

$=\lim_{x\to 1}\frac{5-x-3-x}{(x-1)\left(\sqrt{5-x}+\sqrt{3+x}\right)}$

$=\lim_{x\to 1}\frac{-2(x-1)}{(x-1)\left(\sqrt{5-x}+\sqrt{3+x}\right)}$

$=-\frac{2}{4}$

n. $\lim_{x\to 4}\frac{2-\sqrt{x}}{3-\sqrt{2x+1}}\bullet\frac{3+\sqrt{2x+1}}{3+\sqrt{2x+1}}$

$=\lim_{x\to 4}\frac{\left(2-\sqrt{2}\right)\left(3+\sqrt{2x+1}\right)}{9-2x-1}\bullet\frac{2+\sqrt{x}}{2+\sqrt{x}}$

$=\lim_{x\to 4}\frac{(2-x)\left(3+\sqrt{2x+1}\right)}{4(2-x)\left(2+\sqrt{x}\right)}$

$=\frac{6}{16}$

$=\frac{3}{8}$

o. $\lim_{x\to 0}\frac{2^{2x}-2^{x}}{2^{x}-1}$

$=\lim_{x\to 0}\frac{2^{x}\left(2^{x}-1\right)}{2^{x}-1}$

$=1$

8. a. $\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8}$

Let $u=\sqrt[3]{x}$. Therefore, $u^3=x$ as $x\to 8$, $u\to 2$.

Here, $\lim_{u\to 2}\frac{u-2}{u^3-8}=\lim_{u\to 2}\frac{1}{u^2+2u+4}$

$=\frac{1}{12}$

b. $\lim_{x\to 27}\frac{27-x}{x^{\frac{1}{3}}-3}$

Let $x^{\frac{1}{3}}=u$

$x=u^3$

$x\to 27,\ u\to 3.$

$=\lim_{u\to 3}\frac{u^3-27}{u-3}$

$=-\lim_{u\to 3}\frac{(u-3)\left(u^2+3u+9\right)}{u-3}$

$=-(9+9+9)$

$=-27$

c. $\lim_{x\to 1}\frac{x^{\frac{1}{6}}-1}{x-1}$

$x^{\frac{1}{6}}=u,\ x=u^6$

$x\to 1,\ u\to 1$

$=\lim_{u\to 1}\frac{u-1}{u^6-1}$

$=\lim_{u\to 1}\frac{(u-1)}{(u-1)\left(u^5+u^4+u^3+u^2+u+1\right)}$

$=\frac{1}{6}$

d. $\lim_{x\to 1}\frac{x^{\frac{1}{6}}-1}{x^{\frac{1}{3}}-1}$

Let $x^{\frac{1}{6}}=u$

$u^6=x$

$x^{\frac{1}{3}}=u^2$

As $x\to 1,\ u\to 1$

$=\lim_{u\to 1}\frac{u-1}{u^2-1}$

$=\lim_{u\to 1}\frac{u-1}{(u-1)(u+1)}$

$=\frac{1}{2}$

e. $\lim_{x\to 4}\frac{\sqrt{x}-2}{\sqrt{x^3}-8}$

Let $x^{\frac{1}{2}}=u$

$x^{\frac{3}{2}}=u^3$

$x\to 4,\ u\to 2.$

$=\lim_{u\to 2}\frac{u-2}{u^3-8}$

$=\lim_{u\to 2}\frac{u-2}{(u-2)\left(u^2+2u+4\right)}$

$=\frac{1}{12}$

f. $\lim_{x\to 0}\dfrac{(x+8)^{\frac{1}{3}}-2}{x}$

$=\lim_{u\to 2}\dfrac{u-2}{u^3-8}$

$=\dfrac{1}{12}$

Let $(x+8)^{\frac{1}{3}}=u$

$x+8=u^3$

$x=u^3-8$

$x\to 0,\ u\to 2.$

9. c. $\lim_{x\to 1}\dfrac{x^3+x^2-5x+3}{x^2-2x+1}$

$=\lim_{x\to 1}\dfrac{(x-1)(x-1)(x+3)}{(x-1)(x-1)}$

$=\lim_{x\to 1}(x+3)=4$

d. $\lim_{x\to -1}\dfrac{x^2+x}{x+1}=\lim_{x\to -1}x\dfrac{x(x+1)}{x+1}$

$=-1$

e. $\lim_{x\to 6^+}\dfrac{\sqrt{x^2-5x-6}}{x-3}=0$

f. $\lim_{x\to 0}\dfrac{(2x+1)^{\frac{1}{3}}-1}{x}$

$=\lim_{u\to +1}\dfrac{2(u-1)}{u^3-1}$

$=\lim_{u\to 1}\dfrac{2}{u^2+u+1}$

$=\dfrac{2}{3}$

Let $(2x+1)^{\frac{1}{3}}=u$

$2x+1=u^3$

$x=\dfrac{u^3-1}{2}$

$x\to 0,\ u\to 1.$

g. $\lim_{x\to 2}\dfrac{x^2-4}{\left(\frac{1}{x}\right)-\frac{1}{2}}$

$=\lim_{x\to 2}\dfrac{(x-2)(x+2)}{2-x}\bullet 2x$

$=\lim_{x\to 2}-2x(x+2)$

$=-16$

h. $\lim_{x\to 3}\dfrac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\dfrac{\sqrt{x+1}-2}{(x+1)-4}$

$=\lim_{x\to 3}\dfrac{\sqrt{x+1}-2}{\left(\sqrt{x+1}-2\right)\left(\sqrt{x+1}+2\right)}$

$=\lim_{x\to 3}\dfrac{1}{\sqrt{x+1}+2}$

$=\dfrac{1}{4}$

i. $\lim_{x\to 0}\dfrac{x^2-9x}{5x^3+6x}=\lim_{x\to 0}\dfrac{x-9}{5x^2+6}=-\dfrac{3}{2}$

j. $\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{x+1-1}$

$=\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}$

$=\dfrac{1}{2}$

k. $\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\dfrac{2xh+h^2}{h}$

$=2x$

l. $\lim_{x\to 1}\left(\dfrac{1}{x-1}\right)\left(\dfrac{1}{x+3}-\dfrac{2}{3x+5}\right)$

$=\lim_{x\to 1}\left(\dfrac{1}{x-1}\right)\left(\dfrac{3x+5-2x-6}{(x+3)(3x+5)}\right)$

$=\lim_{x\to 1}\dfrac{1}{(x+3)(3x+5)}$

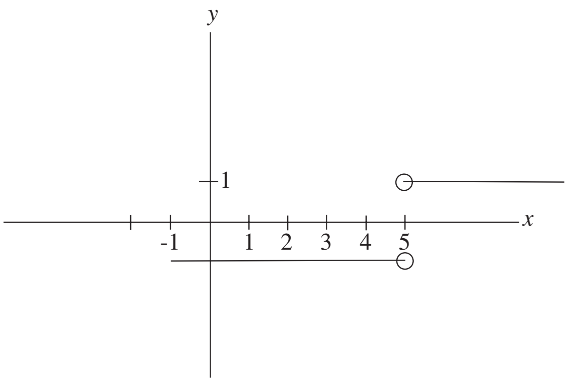
$=\dfrac{1}{4(8)}$

$=\dfrac{1}{32}$

10. a. $\lim_{x\to 5}\frac{|x-5|}{x-5}$ does not exist.

$$\lim_{x\to 5^+}\frac{|x-5|}{x-5}=\lim_{x\to 5^+}\frac{x-5}{x-5}$$
$$=1$$

$$\lim_{x\to 5^-}\frac{|x-5|}{x-5}=\lim_{x\to 5^-}-\left(\frac{x-5}{x-5}\right)$$
$$=-1$$



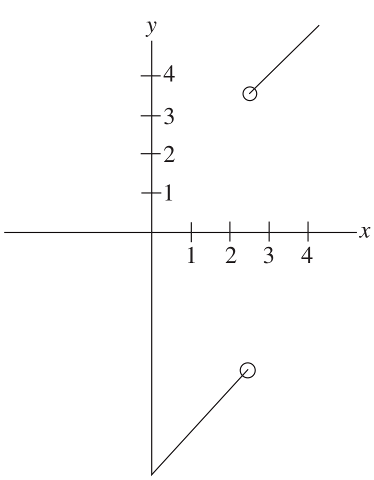
b. $\lim_{x\to\frac{5}{2}}\frac{|2x-5|(x+1)}{2x-5}$ does not exist.

$$|2x-5|=2x-5,\qquad x\geq\frac{5}{2}$$

$$\lim_{x\to\frac{5^+}{2}}\frac{(2x-5)(x+1)}{2x-5}=x+1$$

$$|2x-5|=-(2x-5),\ x<\frac{5}{2}$$

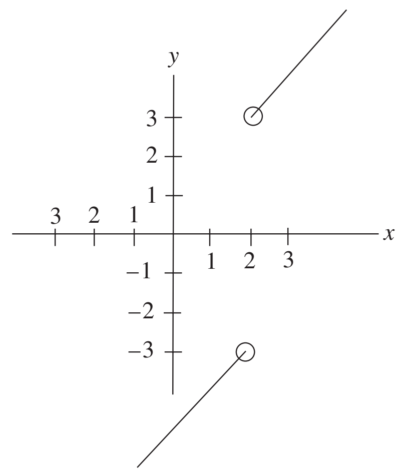
$$\lim_{x\to\frac{5^-}{2}}\frac{-(2x-5)(x+1)}{2x-5}=-(x+1)$$



c. $$\lim_{x\to 2}\frac{x^2-x-2}{|x-2|}=\lim_{x\to 2}\frac{(x-2)(x+1)}{|x-2|}$$

$$\lim_{x\to 2^+}\frac{(x-2)(x+1)}{|x-2|}=\lim_{x\to 2^+}\frac{(x-2)(x+1)}{x-2}=\lim_{x\to 2^+}x+1$$
$$=3$$

$$\lim_{x\to 2^-}\frac{(x-2)(x+1)}{|x-2|}=\lim_{x\to 2^-}-\frac{(x-2)(x+1)}{(x-2)}=\lim_{x\to 2^-}-(x+1)$$
$$=-3$$

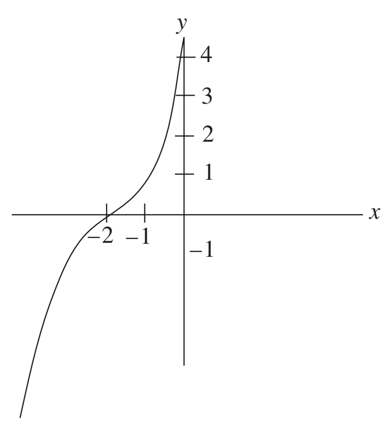


d. $|x+2|=x+2$ if $x>-2$

$=-(x+2)$ if $x<-2$

$$\lim_{x\to -2^+}\frac{(x+2)(x+2)^2}{x+2}=\lim_{x\to -2^+}(x+2)^2=0$$

$$\lim_{x\to -2^-}\frac{(x+2)(x+2)^2}{-(x+2)}=0$$



11. a.

ΔT	T	V	ΔV
	–40	19.1482	
20	–20	20.7908	1.6426
20	0	22.4334	1.6426
	20	24.0760	1.6426
	40	25.7186	1.6426
	60	27.3612	1.6426
	80	29.0038	1.6426

ΔV is constant, therefore T and V form a linear relationship.

b. $$V = \frac{\Delta V}{\Delta T} \bullet T + K$$
$$\frac{\Delta V}{\Delta T} = \frac{1.6426}{20} = 0.08213$$
$$V = 0.08213T + K$$
$$T = 0 \qquad V = 22.4334$$

Therefore, $k = 22.4334$
and $V = 0.08213T + 22.4334$.

c. $T = \dfrac{V - 22.4334}{0.08213}$

d. $\lim\limits_{v \to 0} T = -273.145$

12. $$\lim_{x \to 5} \frac{x^2 - 4}{f(x)}$$
$$= \frac{\lim\limits_{x \to 5}(x^2 - 4)}{\lim\limits_{x \to 5} f(x)}$$
$$= \frac{21}{3}$$
$$= 7$$

13. $\lim\limits_{x \to 4} f(x) = 3$

a. $\lim\limits_{x \to 4} [f(x)]^3 = 3^3 = 27$

b. $$\lim_{x \to 4} \frac{[f(x)]^2 - x^2}{f(x) + x} = \lim_{x \to 4} \frac{(f(x) - x)(f(x) + x)}{f(x) + x}$$
$$= \lim_{x \to 4} (f(x) - x)$$
$$= 3 - 4$$
$$= -1$$

c. $$\lim_{x \to 4} \sqrt{3f(x) - 2x} = \sqrt{3 \times 3 - 2 \times 4}$$
$$= 1$$

14. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$

a. $$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{f(x)}{x} \times x = 0$$

b. $$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x}{g(x)} \frac{f(x)}{x} = 0$$

15. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$ and $\lim\limits_{x \to 0} \dfrac{g(x)}{x} = 2$

a. $$\lim_{x \to 0} f(x) = \lim_{x \to 0} x\left(\frac{f(x)}{x}\right) = 0$$

b. $$\lim_{x \to 0} g(x) = \lim_{x \to 0} x\left(\frac{g(x)}{x}\right) = 0 \times 2$$
$$= 0$$

c. $$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\frac{f(x)}{x}}{\frac{g(x)}{x}} = \frac{1}{2}$$

16.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - \sqrt{2x+1}}{\sqrt{3x+4} - \sqrt{2x+4}}$$
$$= \lim_{x \to 0} \frac{\sqrt{x+1} - \sqrt{2x+1}}{\sqrt{x+1} + \sqrt{2x+1}} \times \frac{\sqrt{x+1} + \sqrt{2x+1}}{\sqrt{3x+4} - \sqrt{2x+4}} \times \frac{\sqrt{3x+4} + \sqrt{2x+4}}{\sqrt{3x+4} + \sqrt{2x+4}}$$
$$= \lim_{x \to 0} \frac{(x+1-2x-1)}{(3x+4-2x-4)} \times \frac{\sqrt{3x+4} + \sqrt{2x+4}}{\sqrt{x+1} + \sqrt{2x+1}}$$
$$= \frac{2+2}{1+1}$$
$$= 2$$

17. $\lim_{x\to 1}\frac{x^2+|x-1|-1}{|x-1|}$

$x\to 1^+ \quad |x-1| = x-1$

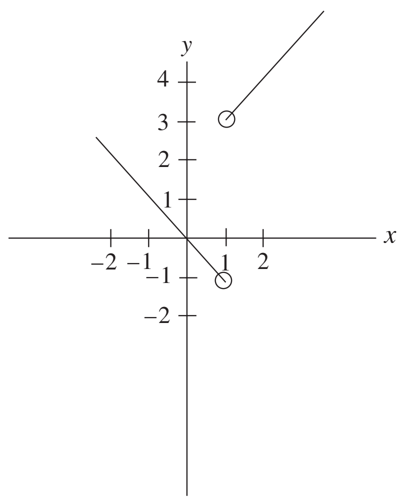
$\therefore \quad \frac{x^2+x-2}{x-1} = \frac{(x+2)(x-1)}{x-1}$

$\lim_{x\to 1^+}\frac{x^2+|x-1|-1}{|x-1|} = 3$

$x\to 1^- \quad |x-1| = -x+1$

$$\lim_{x\to 1^-}\frac{x^2-x}{-x+1} = \lim_{x\to 1^-}\frac{x(x-1)}{-x+1} = -1$$

Therefore, this function does not exist.



18. $\lim_{x\to 1}\frac{x^2+bx-3}{x-1}$

$$x^2+bx-3 = (x-1)(x+3) = x^2+2x-3$$

$$b=2 \quad \lim_{x\to 1}\frac{x^2+2x-3}{x-1} = \lim_{x\to 1}(x+3) = 4$$

Exists for $b = 2$.

19. $\lim_{x\to 0}\frac{\sqrt{mx+b}-3}{x} = 1$

$$\lim_{x\to 0}\frac{\left(\sqrt{mx+b}-3\right)\left(\sqrt{mx+b}+3\right)}{x\left(\sqrt{mx+b}+3\right)} = 1$$

$$\lim_{x\to 0}\frac{mx+b-9}{x\sqrt{mx+b}+3} = 1$$

$$b = 9$$

$$\lim_{x\to 0}\frac{m}{3+3} = 1$$

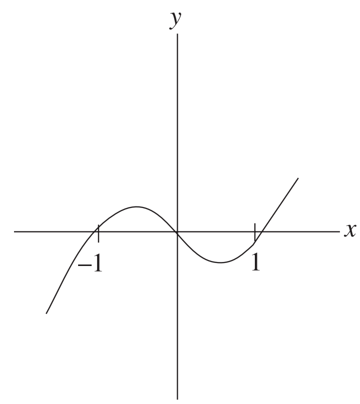
$$\therefore m = 6$$

$$\lim_{x\to 0}\frac{6x}{x\sqrt{6x+9}+3} = \lim_{x\to 0}\frac{6}{\sqrt{6x+9}+3} = 1$$

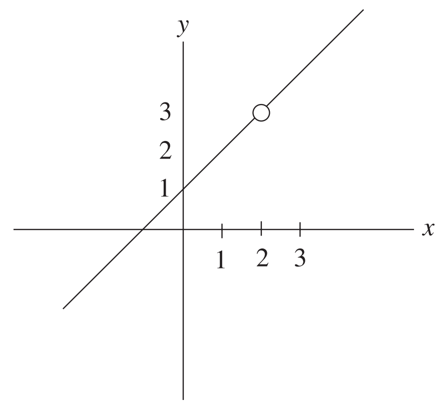
$m = 6, \ b = 9$

Section 3.5

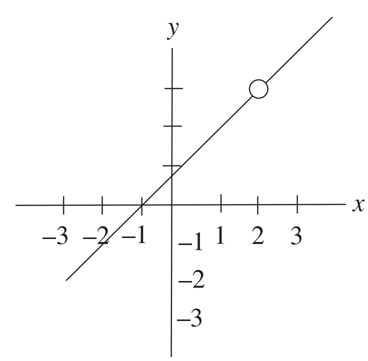
Investigation



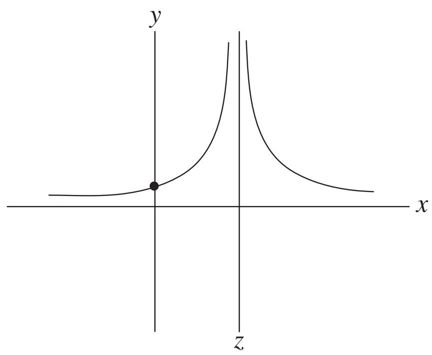
b.



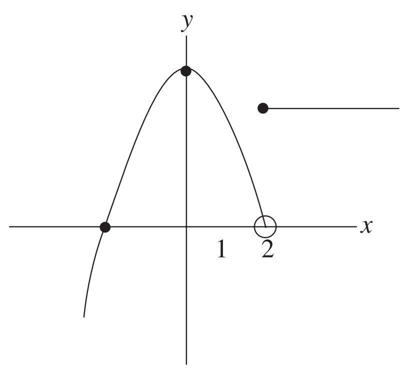
c.



d.



e.



2. **a.** and **c**. are continuous. **b**. contains a hole. **e.** has a jump. **d.** has a vertical asymptote.

3. Window may be too small.

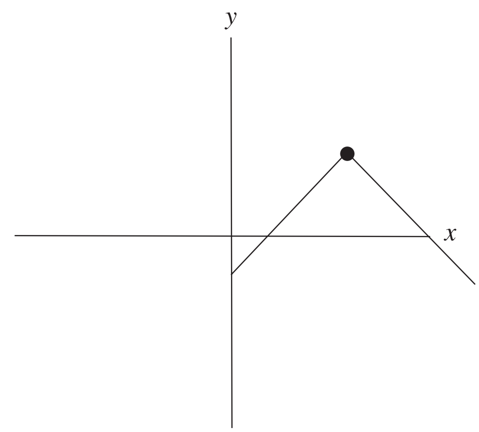
4. Not defined when $x^2 + 300x = 0$
or $x(x+300)=0$.
$x = 0$ or $x = -300$
Continuous for $x < -300$
$-300 < x < 0$
$x > 0$

Exercise 3.5

4. **e.** Discontinuous when

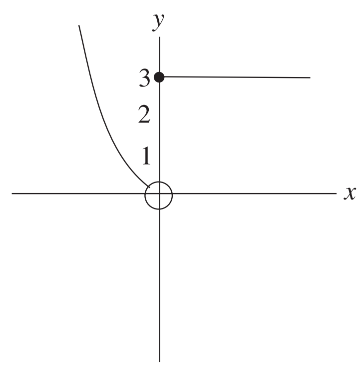
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3 \text{ or } x = 2.$$

7.



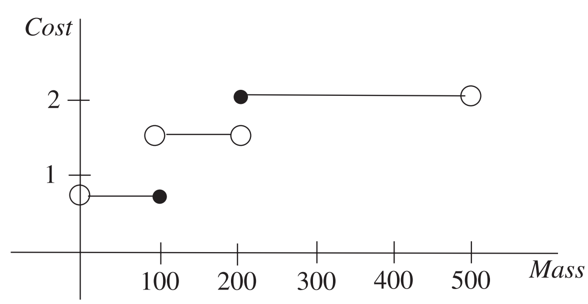
Continuous everywhere.

8.



Discontinuous.

9.



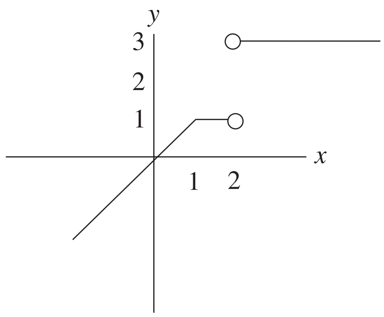
Discontinuous at 0, 100, 200, and 500.

10. $\lim_{x\to 3} f(x) = \lim_{x\to 3} \dfrac{x^2 - x - 6}{x-3}$

$$= \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3}$$
$$= 5$$

Function is discontinuous at $x = 3$.

11.



Discontinuous at $x = 3.0$

12. $g(x)=\begin{cases} x+3, & x\neq 3 \\ 2+\sqrt{k}, & x=3 \end{cases}$

$g(x)$ is continuous.

$\therefore 2+\sqrt{k}=6$

$\sqrt{k}=4,\ k=16$

13. $f(x)=\begin{cases} -x, -3\le x\le -2 \\ ax^2+b, \ -2<x<0 \\ 6, \ x=0 \end{cases}$

at $x=-2,\ 4a+b=2$
at $x=0,\quad b=6$
$\therefore a=-1$

$f(x)=\begin{cases} -x, -3\le x\le -2 \\ -x^2+b, \ -2<x<0 \\ 6, x=0 \end{cases}$

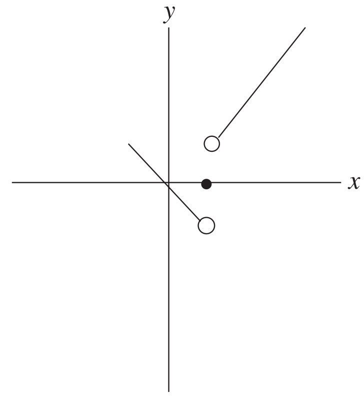
If $a=-1$, $b=6$. $f(x)$ is continuous.

14. $g(x)=\begin{cases} \dfrac{x|x-1|}{x-1}, & x\neq 1 \\ 0, & x=1 \end{cases}$

a. $\left.\begin{array}{l} \lim\limits_{x\to 1^-} g(x)=-1 \\ \lim\limits_{x\to 1^+} g(x)=1 \end{array}\right\} \lim\limits_{x\to 1} g(x)$

$\lim\limits_{x\to 1} g(x)$ does not exist.

b.



Review Exercise

2. a. $f(x)=\dfrac{3}{x+1},\ P(2,1)$

$$m=\lim_{h\to 0}\frac{\dfrac{3}{3+h}-1}{h}$$

$$=\lim_{h\to 0}-\frac{1}{3+h}$$

$$=-\frac{1}{3}$$

b. $g(x)=\sqrt{x+2},\ x=-1$

$$m=\lim_{h\to 0}\frac{\sqrt{-1+h+2}-1}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{x}\times\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{h+1}+1}$$

$$=\frac{1}{2}$$

c. $h(x)=\dfrac{2}{\sqrt{x+5}},\ x=4$

$$m=\lim_{h\to 0}\frac{\dfrac{2}{\sqrt{4+h+5}}-\dfrac{2}{3}}{h}$$

$$=2\lim_{h\to 0}\frac{3-\sqrt{9+h}}{3h\sqrt{9+h}}\times\frac{3+\sqrt{9+h}}{3+\sqrt{9+h}}$$

$$=2\lim_{h\to 0}-\frac{1}{3\sqrt{9+h}\left(3+\sqrt{9+h}\right)}$$

$$=-\frac{2}{9(6)}$$

$$=-\frac{1}{27}$$

d. $f(x)=\dfrac{5}{x-2},\ x=4$

$$m=\lim_{h\to 0}\frac{\frac{5}{4+h-2}-\frac{5}{2}}{h}$$

$$=\lim_{h\to 0}\frac{10-5(2+h)}{h(2+h)(2)}$$

$$=\lim_{h\to 0}-\frac{-5h}{h(2+h)(2)}$$

$$=-\frac{5}{4}$$

3. $f(x)=\begin{cases}4-x^2, & x\le 1\\ 2x+1, & x>1\end{cases}$

a. Slope at $(-1,\ 3)$ $f(x)=4-x^2$

$$m=\lim_{h\to 0}\frac{4-(-1+h)^2-3}{h}$$

$$=\lim_{h\to 0}\frac{4-1+2h-h^2-3}{h}$$

$$=\lim_{h\to 0}(2-h)$$

$$=2$$

Slope of the graph at P (–1, 3) is 2.

b. Slope at P(2, 0.5)

$\therefore f(x)=2x+1$

$$f(2+h)-f(2)=2(2+h)+1-5$$

$$=2h$$

$$m=\lim_{h\to 0}\frac{2h}{h}=2$$

Slope of the graph at P(2, 0.5) is 2.

4. $s(t)=-5t^2+180$

a. $s(0)=180,\ s(1)=175,\ s(2)=160$

Average velocity during the first second is

$$\frac{s(1)-s(0)}{1}=-5\text{ m/s.}$$

Average velocity during the second second is

$$\frac{s(2)-s(1)}{1}=-15\text{ m/s.}$$

b. At $t=4$:

$$s(4+h)-s(4)$$

$$=-5(4+h)^2+180-(-5(16)+180)$$

$$=-80-40h-5h^2+180+80-180$$

$$\frac{s(4+h)-s(4)}{h}=\frac{-40h-5h^2}{h}$$

$$v(4)=\lim_{h\to 0}(-40-5h)=-40$$

Velocity is at –40 m/s.

c. Time to reach ground is when $s(t)=0$.

Therefore, $-5t^2+180=0$

$t^2=36$

$t=6,\ t>0.$

Velocity at $t=6$:

$$s(6+h)=-5(36+12h+h^2)+180$$

$$=-60h-5h^2$$

$$s(6)=0$$

Therefore, $v(6)=\lim_{h\to 0}(-60-5h)=-60.$

5. $M(t)=t^2$ mass in grams

a. Growth during $3\le t\le 3.01$

$M(3.01)=(3.01)^2=9.0601$

$M(3)=3^2\quad =9$

Grew 0.0601g during this time interval.

b. Average rate of growth is

$$\frac{0.0601}{0.01}=6.01\text{ g/s}.$$

c. $s(3+h)=9+6h+h^2$

$s(3)=9$

$$\frac{s(3+h)-s(3)}{h}=\frac{6h+h^2}{h}$$

Rate of growth is $\lim_{h\to 0}(6+h)=6$ g/s.

6. $Q(t)=10^4(t^2+15t+70)$ tonnes of waste, $0\le t\le 10$

a. At $t=0$,

$Q(t)=70\times 10^4$

$=700\ 000.$

700 000 t have accumulated up to now.

b. Over the next three years, the average rate of change:

$$Q(3) = 10^4(9+45+70)$$
$$= 124 \times 10^4$$
$$Q(0) = 70 \times 10^4$$
$$\frac{Q(3)-Q(0)}{3} = \frac{5^4 \times 16^4}{3}$$
$$= 18 \times 10^4 \text{ t per year.}$$

c. Present rate of change:

$$Q(h) = 10^4(h^2 + 15h + 70)$$
$$Q(0) = 10^4 + 70$$
$$\lim_{h\to 0}\frac{Q(h)-Q(0)}{h} = \lim_{h\to 0} 10^4(h+15)$$
$$= 15 \times 10^4 \text{ t per year.}$$

d. $Q(a+h) = 10^4\left[a^2 + 2ah + h^2 + 15a + 15h + 70\right]$

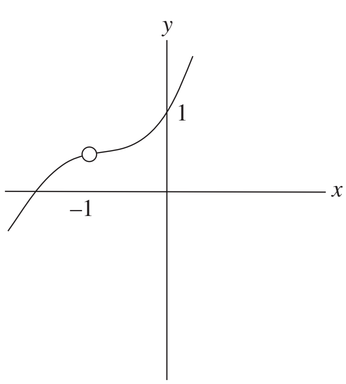
$$Q(a) = 10^4\left[a^2 + 15a + 70\right]$$
$$\frac{Q(a+h)-Q(a)}{h} = \frac{10^4\left[2ah + h^2 + 15h\right]}{h}$$
$$\lim_{h\to 0}\frac{Q(a+h)-Q(a)}{h} = \lim_{h\to 0} 10^4(2a+h+15)$$
$$= (2a+15)10^4$$

Now,

$$(2a+15)10^4 = 3 \times 10^5$$
$$2a + 15 = 30$$
$$a = 7.5.$$

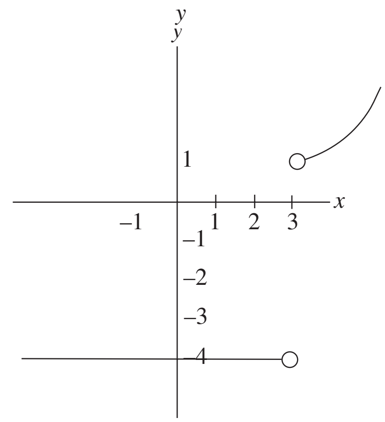
It will take 7.5 years to reach a rate of 3.0×10^5 t per year.

8. **a.** $\lim_{x\to -1} f(x) = 0.5$, f is discontinuous at $x = -1$



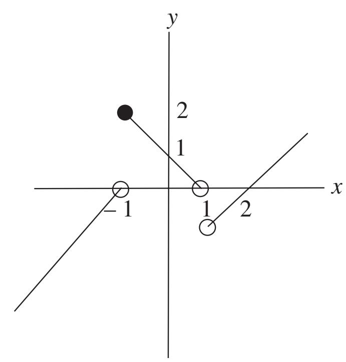
b. $f(x) = -4$ if $x < 3$; f is increasing for $x > 3$

$\lim_{x\to 3^+} f(x) = 1$



9. $f(x) = \begin{cases} x+1, & x < -1 \\ -x+1, & -1 \le x < 1 \\ x-2, & x > 1 \end{cases}$

a.



Discontinuous at $x = -1$ and $x = 1$.

b. They do not exist.

10. $f(x) = \frac{x^2 - x - 6}{x-3}$

$$= \frac{(x-3)(x+2)}{(x-3)}$$
$$\lim_{x\to 3} f(x) = \lim_{x\to 3}(x+2)$$
$$= 5$$

$f(x)$ is not continuous at $x = 3$.

11. $$f(x)=\frac{2x-2}{x^2+x-2}$$
$$=\frac{2(x-1)}{(x-1)(x+2)}$$

a. f is discontinuous at $x=1$ and $x=-2$.

b. $$\lim_{x\to1}f(x)=\lim_{x\to1}\frac{2}{x+2}$$
$$=\frac{2}{3}$$

$$\lim_{x\to-2}f(x):\ \lim_{x\to-2^+}\frac{2}{x+2}=+\infty$$
$$\lim_{x\to-2^-}\frac{2}{x+2}=-\infty$$

$\lim_{x\to-2}f(x)$ does not exist.

12. a. $f(x)=\frac{1}{x^2}$, $\lim_{x\to0}f(x)$ does not exist.

b. $g(x)=x(x-5)$, $\lim_{x\to0}g(x)=0$

c. $$h(x)=\frac{x^3-27}{x^2-9},$$
$$\lim_{x\to4}h(x)=\frac{37}{7}=5.2857$$

$\lim_{x\to-3}h(x)$ does not exist.

15. a. $$f(x)=\frac{\sqrt{x+2}-2}{x-2}$$

x	2.1	2.01	2.001
$f(x)$	0.24846	0.24984	0.24998

$x = 2.0001$
$f(x) - 0.25$

b. $\lim_{x\to2}f(x)=0.25$

c. $$\lim_{x\to2}\frac{\sqrt{x+2}-2}{x-2}\times\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}$$
$$=\lim_{x\to2}\frac{1}{\sqrt{x+2}+2}$$
$$=\frac{1}{4}=0.25$$

16. a. $$\lim_{h\to0}\frac{(5+h)^2-25}{h}$$
$$=\lim_{h\to0}(10+h)$$
$$=10.$$

Slope of the tangent to $y=x^2$ at $x=5$ is 10.

b. $$\lim_{h\to0}\frac{\sqrt{4+h}-2}{h}=\lim_{h\to0}\frac{\sqrt{4+h}-2}{4+h-4}$$
$$=\lim_{h\to0}\frac{1}{\sqrt{4+h}+2}$$
$$=\frac{1}{4}$$

Slope of the tangent to $y=\sqrt{x}$ at $x=4$ is $\frac{1}{4}$.

c. $$\lim_{h\to0}\frac{\frac{1}{4+h}-\frac{1}{4}}{h}=\lim_{h\to0}\frac{4-4h}{4(4+h)(h)}$$
$$=\lim_{h\to0}-\frac{1}{4(4+h)}$$
$$=-\frac{1}{16}$$

Slope of the tangent to $y=\frac{1}{x}$ at ($x = 4$) is $-\frac{1}{16}$.

d. $$\lim_{h\to0}\frac{(343+h)^{\frac{1}{3}}-7}{h}=\lim_{h\to0}\frac{(343+h)^{\frac{1}{3}}-7}{343+h-343}$$
$$=\lim_{h\to0}\frac{(343+h)^{\frac{1}{3}}-7}{\left((343+h)^{\frac{1}{3}}-7\right)\left((343+h)^{\frac{2}{3}}+7(343+h)^{\frac{1}{3}}+49\right)}$$
$$=\frac{1}{49+49+49}$$
$$=\frac{1}{147}$$

Slope of the tangent to $y=x^{\frac{1}{3}}$ at $x=343$ is $\frac{1}{147}$.

17. h. $$\lim_{x\to a}\frac{(x+4a)^2-25a^2}{x-a}=\lim_{x\to a}\frac{(x-a)(x+9a)}{x-a}$$
$$=10a$$

o. $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5-x}}{x}\times\frac{\sqrt{x+5}+\sqrt{5-x}}{\sqrt{x+5}+\sqrt{5-x}}$

$=\lim_{x\to 0}\frac{x+5-5+x}{x\left(\sqrt{x+5}+\sqrt{5-x}\right)}$

$=\frac{1}{2\sqrt{5}}$

q. $\lim_{x\to -2}\frac{x^3+x^2-8x-12}{x+2}$

$=\lim_{x\to -2}\frac{(x+2)(x^2-1x-6)}{(x+2)}$

$=4+2-6$

$=0$

r. $\lim_{x\to 2}\frac{x^3+x^2-12}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^2+3x+6)}{x-2}$

$=4+6+6$

$=16$

t. $\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{2+x}-\frac{1}{2}\right)$

$=\lim_{x\to 0}\frac{1}{x}\times-\frac{x}{2(2+x)}$

$=\lim_{x\to 0}-\frac{1}{2(2+x)}$

$=-\frac{1}{4}$

u. $\lim_{x\to -1}\frac{108(x^2+2x)(x+1)(x+1)(x+1)}{(x+1)^3(x^2-x+1)^3(x-1)}$

$=\lim_{x\to -1}\frac{108(x^2+2x)}{(x^2-x+1)^3(x-1)}$

$=-\frac{108}{27(-2)}$

$=2$

18. d. $\lim_{x\to 0}\frac{|x|}{x}$

$x\to 0^-\quad |x|=-x$

$\lim_{x\to 0^-}\frac{|x|}{x}=-1$

$\lim_{x\to 0^+}\frac{|x|}{x}=1$

$\lim_{x\to 0^+}\frac{|x|}{x}\neq\lim_{x\to 0^-}\frac{|x|}{x}$

e. $f(x)=\begin{cases}-5, & x<1\\ 2, & x\geq 1\end{cases}$

$\lim_{x\to 1^-}f(x)=-5\neq\lim_{x\to 1^+}f(x)=2$

f. $f(x)=\begin{cases}5x^2, & x<-1\\ 2x+1, & x\geq -1\end{cases}$

$\lim_{x\to -1^+}f(x)=-1$

$\lim_{x\to -1^-}f(x)=5$

$\lim_{x\to -1^+}f(x)\neq\lim_{x\to -1^-}f(x)$

Therefore, $\lim_{x\to -1}f(x)$ does not exist.

Chapter 3 Test

3. $\lim_{x\to 1}\frac{1}{x-1}$ does not exist since

$\lim_{x\to 1^+}\frac{1}{x-1}=+\infty\neq\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$.

4. $f(x)=\frac{x}{x-3},\quad g(x)=\frac{3}{x-3}$

$\lim_{x\to 3}f(x)$ does not exist.

$\lim_{x\to 3}g(x)$ does not exist.

$\lim_{x\to 3}[f(x)-g(x)]=\lim_{x\to 3}\frac{x-3}{x-3}$

$=\lim_{x\to 3}1$

$=1$

5. $f(x) = 5x^2 - 8x$

$f(-2) = 5(4) - 8(-2) = 20 + 16 = 36$

$f(1) = 5 - 8 = -3$

Slope of secant is $\dfrac{36+3}{-2-1} = -\dfrac{39}{3}$
$= -13.$

6. Slope of a line perpendicular to $y = \dfrac{3}{4}x + 5$ is $-\dfrac{4}{5}$.

7. For $f(x) = \dfrac{\sqrt{x^2+100}}{5}$, y-intercept is 2.

8. Through (0, –2), slope –1,
$y = -x - 2$ or $x + y + 2 = 0$.

9. **a.** $\lim\limits_{x\to 1} f(x)$ does not exist.

b. $\lim\limits_{x\to 2} f(x) = 1$

c. $\lim\limits_{x\to 4^-} f(x) = 1$

d. f is discontinuous at $x = 1$ and $x = 2$.

10. $P = 100\ 000 + 4000t$
P – population
t – years

a. $t = 20$ $\quad P = 100\ 000 + 80\ 000$
$= 180\ 000$
Population in 20 years will be 180 000 people.

b. $P(a+h) - P(a)$
$= (100\ 000 + 4000(a+h)) - (100\ 000 + 4000a)$
$= 4000h$
Growth rate:
$$\lim_{h\to 0} \frac{P(a+h) - Pa}{h} = 4000$$
Growth rate is 4000 people per year.

11. **a.** Average velocity from $t = 2$ to $t = 5$:
$$\frac{s(5) - s(2)}{3} = \frac{(40-25)-(16-4)}{3}$$
$$= \frac{15-12}{3}$$
$$= 1$$
Average velocity from $t = 2$ to $t = 5$ is 1 km/h.

b. $s(3+h) - s(3)$
$= 8(3+h) - (3+h)^2 - (24-9)$
$= 24 + 8h - 9 - 6h - h^2 - 15$
$= 2h - h^2$
$$v(3) = \lim_{h\to 0} \frac{2h - h^2}{h} = 2$$
Velocity at $t = 3$ is 2 km/h.

12. $f(x) = \sqrt{x+11}$,
Average rate of change from
$x = 5$ to $x = 5 + h$:
$$\frac{f(5+h) - f(5)}{h}$$
$$= \frac{\sqrt{16+h} - \sqrt{16}}{h}$$

13. $f(x) = \dfrac{x}{x^2 - 15}$

Slope of the tangent at $x = 4$:

$$f(4+h) = \frac{4+h}{(4+h)^2 - 15}$$
$$= \frac{4+h}{1 + 8h + h^2}$$
$$f(4) = \frac{4}{1}$$
$$f(4+h) - f(4) = \frac{4+h}{1+8h+h^2} - 4$$
$$= \frac{4 + h - 4 - 32h - 4h^2}{1 + 2h + h^2}$$
$$= -\frac{31h - 4h^2}{(1 + 2h + h^2)}$$
$$\lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{(-31 - 4h)}{1 + 2h + h^2}$$
$$= -31$$
Slope of the tangent at $x = 4$ is –31.

14. **a.** $\lim\limits_{x\to 3} \dfrac{4x^2 - 36}{2x - 6} = \lim\limits_{x\to 3} \dfrac{2(x-3)(x+3)}{(x-3)}$
$= 12$

b. $\lim\limits_{x\to 2} \dfrac{2x^2 - x - 6}{3x^2 - 7x + 2} = \lim\limits_{x\to 2} \dfrac{(2x+3)(x-2)}{(x-2)(3x-1)}$
$= \dfrac{7}{5}$

c. $\lim\limits_{x\to 5}\dfrac{x-5}{\sqrt{x-1}-2}=\lim\limits_{x\to 5}\dfrac{(x-1)-4}{\sqrt{x-1}-2}$

$$=\lim_{x\to 5}\frac{\left(\sqrt{x-1}-2\right)\left(\sqrt{x-1}+2\right)}{\sqrt{x-1}-2}$$

$$=4$$

d. $\lim\limits_{x\to -1}\dfrac{x^3+1}{x^4-1}=\lim\limits_{x\to -1}\dfrac{(x+1)(x^2-x+1)}{(x-1)(x+1)(x^2+1)}$

$$=\frac{3}{-2(2)}$$

$$=-\frac{3}{4}$$

e. $\lim\limits_{x\to 3}\left(\dfrac{1}{x-3}-\dfrac{6}{x^2-9}\right)=\lim\limits_{x\to 3}\dfrac{(x+3)-6}{(x-3)(x+3)}$

$$=\lim_{x\to 3}\frac{1}{x+3}$$

$$=\frac{1}{6}$$

f. $\lim\limits_{x\to 0}\dfrac{(x+8)^{\frac{1}{3}}-2}{x}=\lim\limits_{x\to 0}\dfrac{(x+8)^{\frac{1}{3}}-2}{(x+8)-8}$

$$=\lim_{x\to 0}\frac{(x+8)^{\frac{1}{3}}-2}{\left((x+8)^{\frac{1}{3}}-2\right)\left((x+8)^{\frac{2}{3}}+2(x+8)^{\frac{1}{3}}+4\right)}$$

$$=\frac{1}{4+4+4}$$

$$=\frac{1}{12}$$

15. $f(x)=\begin{cases} ax+3, \ x>5 \\ 8, \ x=5 \\ x^2+bx+a, \ x<5 \end{cases}$

$f(x)$ is continuous.

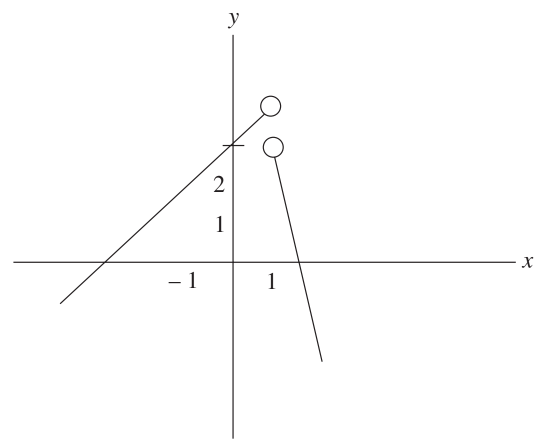
Therefore, $5a+3=8$ $\qquad a=1$

$25+5b+a=8$ $\qquad 5b=-18$

$$b=-\frac{18}{5}$$

16. a. $f(0)=3$ **b.** $\lim\limits_{x\to 1^+}f(x)=3$

c. $\lim\limits_{x\to 1^-}f(x)=4$ **d.** $f(2)=-1$



17. $f(x)=\begin{cases}\dfrac{x-2}{\sqrt{7x+2}-\sqrt{6x+4}}, & x\ge -\dfrac{2}{7}, x\ne 2 \\ k, \ x=2 \end{cases}$

$$\frac{x-2}{\sqrt{7x+2}-\sqrt{6x+4}}\times\frac{\sqrt{7x+2}+\sqrt{6x+4}}{\sqrt{7x+2}+\sqrt{6x+4}}$$

$$=\frac{(x-2)\left(\sqrt{7x+2}+\sqrt{6x+4}\right)}{7x+2-6x-4}$$

$$=\sqrt{7x+2}+\sqrt{6x+4}$$

Now, when $x=2$,

$k=\sqrt{7(2)+2}+\sqrt{6(2)+4}$

$=4+4$

$k=8.$